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On the functional-differential equation of advanced type $f'(x) = af(2x)$ with $f(0) = 0$

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Abstract

In this paper we construct solutions for the equation

$$\begin{cases} f'(x) = af(2x), & -\infty < x < +\infty, \\ f(0) = 0, \end{cases}$$

where a is a constant with $a \neq 0$. The solutions are infinitely differentiable and bounded on $\mathbb{R}$. Using our method, we can get numerical data easily with a computer. Applying one of the solutions we show that the derivative of order $k \geqslant 0$ of a function $v \in C^k(\mathbb{R})$ or $v \in L_k^p(\mathbb{R})$ coincides $\lim_{\epsilon \to 0} v * G_{k,\epsilon}$, where $L_k^p(\mathbb{R})$ is the Sobolev space and $\{G_{k,\epsilon}\}_{\epsilon > 0}$ is a family of C^∞-functions.
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1. Introduction

Frederickson [1,2] (1971) investigated functional-differential equations of advanced type

$$f'(x) = af(\lambda x) + bf(x), \tag{1}$$

where $\lambda > 1$, and provided several properties of solutions. Kato and McLeod [4] (1971) and Kato [3] (1972) studied asymptotic behavior of solutions of (1).

Frederickson [1] provided a global existence theorem for equations

$$f'(x) = F\big(f(2x)\big), \quad x \in \mathbb{R} = (-\infty, +\infty),$$

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where F is an odd, continuous function with $F(s) > 0$ for $s > 0$. In [1] he applied the Schauder fixed point theorem to the proof. He showed that the absolute value of the solution $|f(x)|$ is periodic for $x \geqslant 0$. Frederickson [2] also provided a constructive method of solutions for equations of advanced type

$$f'(z) = af(\lambda z) + bf(z),$$

where $a, b \in \mathbb{C}$ and $\lambda > 1$. He gave solutions in the form of a Dirichlet series

$$\varphi(z, \beta) = \sum_{n\in\mathbb{Z}} c_n e^{\beta\lambda^n z}, \qquad \Re(\beta z) \leqslant 0,$$

where β is allowed to vary as a parameter. In the case of $b = 0$ and $\beta = i$, the solution is analytic in the upper half plane $\Im z > 0$, continuous on $\Im z \geqslant 0$, and the line $\Im z = 0$ is a natural boundary.

In this paper, using another method, we construct solutions for the equation

$$\begin{cases} f'(x) = af(2x), & x \in \mathbb{R} = (-\infty, +\infty), \\ f(0) = 0, \end{cases} \tag{2}$$

where a is a constant with $a \neq 0$. The solution is not unique. If f is a solution, then a constant times f is also a solution. Our solutions are infinitely differentiable and bounded on $\mathbb{R}$. Using our method we can get numerical data easily with a computer. We also give the Fourier transform of one of our solutions and show the uniqueness of the solution of (2) with a certain condition. Applying one of the solutions we show that the derivative of order $k \geqslant 0$ of a function $v \in C^k(\mathbb{R})$ or $v \in L^p_k(\mathbb{R})$ coincides $\lim_{\epsilon\to 0} v * G_{k,\epsilon}$, where $L^p_k(\mathbb{R})$ is the Sobolev space and $\{G_{k,\epsilon}\}_{\epsilon>0}$ is a family of C^∞-functions.

2. Main results

Our main results are the following:

Theorem 1. *If $a > 0$, then there exists a solution $f \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ of (2) with $f \not\equiv 0$ such that f vanishes on $(-\infty, 0]$ and $|f|$ has period $4/a$ on $[0, +\infty)$. If $a < 0$, then there exists a solution $f \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ of (2) with $f \not\equiv 0$ such that f vanishes on $[0, +\infty)$ and $|f|$ has period $4/|a|$ on $(-\infty, 0]$.*

If there exists a solution f of

$$\begin{cases} f'(x) = 4f(2x), & x \in \mathbb{R}, \\ f(0) = 0, \end{cases} \tag{3}$$

such that f vanishes on $(-\infty, 0]$ and $|f|$ has period 1 on $[0, +\infty)$, then $f(ax/4)$ is a solution of (2). If $a > 0$, then $f(ax/4)$ vanishes on $(-\infty, 0]$ and $|f(ax/4)|$ has period $4/a$ on $[0, +\infty)$. If $a < 0$, then $f(ax/4)$ vanishes on $[0, +\infty)$ and $|f(ax/4)|$ has period $4/|a|$ on $(-\infty, 0]$.

To construct a solution of (3), we use the function u in the next lemma.

Lemma 2. *There exist a nonnegative function u with* $\operatorname{supp} u \subset [0, 1]$ *such that*

(i) $u \in C^\infty(\mathbb{R})$,
(ii) $u(x)$ *satisfies* (3) *for* $0 \leqslant x \leqslant 1/2$,
(iii) $u(x) = u(1 - x)$ *for* $0 \leqslant x \leqslant 1$,

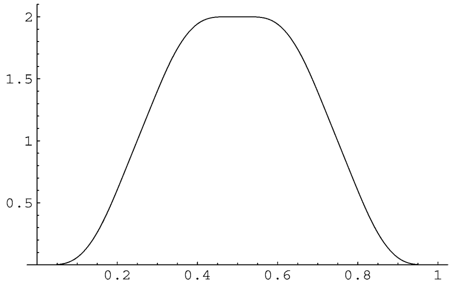

Fig. 1. $u(x)$.

(iv) $u(x) + u(1/2 - x) = 2$ *for* $0 \leqslant x \leqslant 1/2$,
(v) $u(x)$ *is increase for* $0 \leqslant x \leqslant 1/2$,
(vi) $u^{(k)}(0) = u^{(k)}(1) = 0$ *for* $k = 0, 1, 2, \ldots,$ *and*
(vii) $\int_0^1 u(x)\,dx = 1$.

Remark 3. The function u in Lemma 2 is unique. If v satisfies (ii), (iii) and $v(1/4) = b$ ($b \in \mathbb{R}$), then $v(x) = bu(x)$ for $0 \leqslant x \leqslant 1$ (see Theorem 11 and Remark 12).

The graph of the function $u(x)$ is in Fig. 1.

Theorem 4. *A solution* $f(x)$ *of* (3) *is expressed by*

$$f(x) = \sum_{k=1}^{\infty} (-1)^{n_k} u(x - k + 1), \tag{4}$$

where u *is in Lemma* 2 *and*

$$\begin{cases} n_1 = 0, & n_2 = 1, \\ n_{2k-1} = 1,\ n_{2k} = 0, & \text{if } n_k = 1 \ (k \geqslant 2), \\ n_{2k-1} = 0,\ n_{2k} = 1, & \text{if } n_k = 0 \ (k \geqslant 2). \end{cases}$$

Remark 5. Remark 3 shows that, if $g(x)$ is a solution of (3) with $g(x) = g(1 - x)$ for $0 \leqslant x \leqslant 1$ and $g(1/4) = b$, then $g(x) = bf(x)$ for $0 \leqslant x \leqslant 1$, where f is in (4). If g is a solution of (3), then the value of $g(x)$ on $[2^k, 2^{k+1}]$ is determined by the value of $g'(x)$ on $[2^{k-1}, 2^k]$, $k = 0, 1, 2, \ldots.$ Therefore, $g(x) = bf(x)$ for $x \geqslant 0$.

The solution $f(x)$ satisfies

$$f\left(2^{n-1} + t\right) = (-1)^n f\left(2^{n-1} - t\right), \quad t \in \left[0, 2^{n-1}\right],\ n = 1, 2, \ldots, \tag{5}$$

and $|f(x)|$ has period 1 on $[0, +\infty)$ as Frederickson mentioned in [1]. Actually, $f(x) = f(1-x)$ ($x \in [0, 1/2]$) implies $f(1/2 + t) = f(1/2 - t)$ ($t \in [0, 1/2]$). Then $4f(1 + 2t) = f'(1/2 + t) = -f'(1/2 - t) = -4f(1 - 2t)$ ($t \in [0, 1/2]$), i.e. $f(1 + t) = -f(1 - t)$ ($t \in [0, 1]$). This is the equality (5) for $n = 1$. In the same way we have (5) for $n = 2, 3, \ldots.$ Moreover, $f \in C^{\infty}(\mathbb{R})$. The graph of the solution $f(x)$ of (3) is in Fig. 2.

The next theorem is another expression of the solution f. Let

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}\,dx,$$

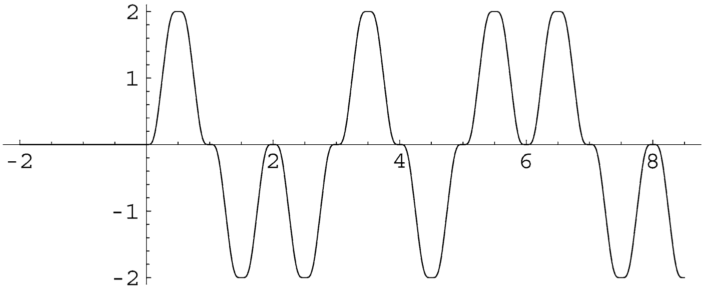

Fig. 2. The solution $f(x)$ of (3).

and $\operatorname{sinc}\xi = \sin(\pi\xi)/(\pi\xi)$. Let $\mathcal{S}$ be the space of all rapidly decreasing functions and $\mathcal{S}'$ be the space of all tempered distributions.

Theorem 6. *The Fourier transform of the solution f in* (4) *is expressed by*

$$\hat{f}(\xi) = \sum_{k=1}^{\infty} (-1)^{n_k} e^{-i(k-1)\xi} \hat{u}(\xi), \quad \textit{in } \mathcal{S}',$$

where

$$\hat{u}(\xi) = e^{-i\xi/2} \lim_{n\to\infty} \prod_{k=1}^{n} \operatorname{sinc}\bigl(\xi/(2^{k+1}\pi)\bigr) \cdot \operatorname{sinc}\bigl(\xi/(2^{n+1}\pi)\bigr), \quad \textit{uniformly on } \mathbb{R}.$$

For a nonnegative integer k and $1 \leqslant p \leqslant \infty$, let $L_k^p(\mathbb{R})$ be the Sobolev space. Applying f in (4), we have the following.

Theorem 7. *Let f be the solution in* (4) *and*

$$G_{k,\epsilon}(x) = \bigl(2^{k(k-1)/2}\epsilon^{k+1}\bigr)^{-1} \bigl(f\chi_{[0,2^k]}\bigr)(x/\epsilon).$$

If $v \in C^k(\mathbb{R})$ or $v \in L_k^p(\mathbb{R})$ $(k \geqslant 0,\ 1 \leqslant p < \infty)$, then

$$\frac{d^k v}{dx^k} = \lim_{\epsilon\to 0} v * G_{k,\epsilon},$$

uniformly on each compact subset in $\mathbb{R}$ or in $L^p(\mathbb{R})$, respectively.

Remark 8. $G_{k,\epsilon}$ is in $C^\infty(\mathbb{R})$ with compact support. To prove the theorem, we use Friedrichs' molifier $\frac{d^k v}{dx^k} * u_\delta = v * \frac{d^k u_\delta}{dx^k}$, where $u_\delta = u(x/\delta)/\delta$, $\delta > 0$, and u is the function in Lemma 2.

We construct a solution of (3) and prove Lemma 2 and Theorem 4 in the next section. We prove Theorems 6 and 7 in Sections 4 and 5, respectively.

3. Construction of a solution of (3)

In this section we prove Lemma 2 and Theorem 4 to construct a solution of (3). The initial value problem (3) is equivalent to the integral equation

$$f(x)=2\int_0^{2x} f(t)\,dt, \quad x\in\mathbb{R}. \tag{6}$$

If f is a solution of (3) and satisfies $f(x)=f(1-x)$ $(0\leqslant x\leqslant 1)$, then we have

$$\begin{aligned}2\int_0^{2x} f(t)\,dt &= f(x)=f(1-x)\\ &= 2\int_0^{2(1-x)} f(t)\,dt = 2\int_0^{2(1-x)} f(1-t)\,dt = 2\int_{2x-1}^{1} f(t)\,dt, \quad \frac{1}{2}\leqslant x\leqslant 1.\end{aligned}$$

Then we define a function space X and an operator $T: X\to X$ as follows:

$$X=\bigl\{u\in L^1(\mathbb{R})\colon\ \operatorname{supp} u\subset[0,1],\ u(x)=u(1-x)\bigr\},$$
$$Tu(x)=\begin{cases} 2\int_0^{2x} u(t)\,dt, & x\in[0,1/2],\\ 2\int_{2x-1}^{1} u(t)\,dt, & x\in(1/2,1],\\ 0, & x\notin[0,1].\end{cases}$$

We construct a function $u\in X$ such that $u=Tu$ in the proof of Lemma 2. Then $u(x)$ satisfies (6) for $x\in[0,1/2]$. The function $u(x)$ satisfies (6) for $x\in(-\infty,0]$ clearly. In the proof of Theorem 4 the function $u(x)$ is extended uniquely to the right (increasing value of x) by using the equality $f'(x)=4f(2x)$.

3.1. *Proof of Lemma 2 (Step 1)*

We state two lemmas. Let χ_I be the characteristic function of the interval $I\subset\mathbb{R}$.

Lemma 9. *The operator T is expressed by*

$$Tu(x)=2(\chi_{[0,1]}*u)(2x), \quad u\in X, \tag{7}$$

and satisfies

$$\int_0^1 u(x)\,dx=\int_0^1 Tu(x)\,dx, \tag{8}$$

$$Tu(x)\geqslant 0 \quad \text{if } u(x)\geqslant 0, \tag{9}$$

$$\bigl|Tu(x)\bigr|\leqslant 2\int_0^1 \bigl|u(x)\bigr|\,dx. \tag{10}$$

Proof. From $\operatorname{supp} u \subset [0,1]$ and $\chi_{[0,1]}(2x-t) = \chi_{[2x-1,2x]}(t)$ it follows that

$$2(\chi_{[0,1]} * u)(2x) = 2\int_0^1 \chi_{[0,1]}(2x-t)u(t)\,dt = 2\int_0^1 \chi_{[2x-1,2x]}(t)u(t)\,dt.$$

Then we have (7). The properties (8)–(10) follows from (7) and the definition of T. $\square$

Let $u_0 = \chi_{[0,1]} \in X$ and $u_{n+1} = Tu_n$, $n = 0,1,2\ldots.$ Then $\{u_n\}_{n=0}^{+\infty}$ is a sequence of functions in X. It follows from Lemma 9 that

$$\int_0^1 u_n(x)\,dx = 1, \quad 0 \leqslant u_n(x) \leqslant 2,\ x \in \mathbb{R}. \tag{11}$$

We note that

$$Tu_n(x) + Tu_n(1/2 - x) = 2 \quad x \in [0,1/2],\ n = 0,1,2,\ldots. \tag{12}$$

Actually,

$$2\int_0^{2x} u_n(t)\,dt + 2\int_0^{1-2x} u_n(t)\,dt = 2\int_0^{2x} u_n(t)\,dt + 2\int_{2x}^1 u_n(t)\,dt = 2.$$

Lemma 10. *There exists a function $u \in X$ such that $u_n(x) \to u(x)$ as $n \to +\infty$ uniformly on $\mathbb{R}$.*

Proof. Let $g_n = u_n - u_{n-1}$, $n = 1,2,\ldots.$ Then $g_{n+1} = Tg_n$ and

$$\begin{cases} g_n(x) + g_n(1/2 - x) = 0, & x \in [0,1/2], \\ g_n(x) = g_n(1-x), & x \in [0,1], \\ g_n(x) + g_n(3/2 - x) = 0, & x \in [1/2,1], \end{cases} \qquad n = 1,2,\ldots. \tag{13}$$

For example,

$$\begin{aligned} g_1 &= u_1 - u_0 = \big(4x\chi_{[0,1/2]} + (-4x+4)\chi_{(1/2,1]}\big) - \chi_{[0,1]} \\ &= (4x-1)\chi_{[0,1/2]} + (-4x+3)\chi_{(1/2,1]}, \\ g_2 &= Tg_1 = \big(16(x-1/8)^2 - 1/4\big)\chi_{[0,1/4]} + \big(-16(x-3/8)^2 + 1/4\big)\chi_{(1/4,1/2]} \\ &\quad + \big(-16(x-5/8)^2 + 1/4\big)\chi_{(1/2,3/4]} + \big(16(x-7/8)^2 - 1/4\big)\chi_{(3/4,1]}. \end{aligned}$$

We show

$$\begin{cases} g_n(x) \leqslant 0, & x \in [0,1/4] \cup [3/4,1], \\ g_n(x) \geqslant 0, & x \in [1/4,3/4], \end{cases} \qquad n = 1,2,\ldots, \tag{14}$$

by induction. At first, g_1 satisfies (14) clearly. Assume that g_n satisfies (14). From $g_n(x) \leqslant 0$, $x \in [0,1/4]$, and $g_n(x) + g_n(1/2 - x) = 0$, $x \in [0,1/2]$, it follows that

$$g_{n+1}(x) = Tg_n(x) = 2\int_0^{2x} g_n(t)\,dt \leqslant 0, \quad x \in [0,1/8],$$

and

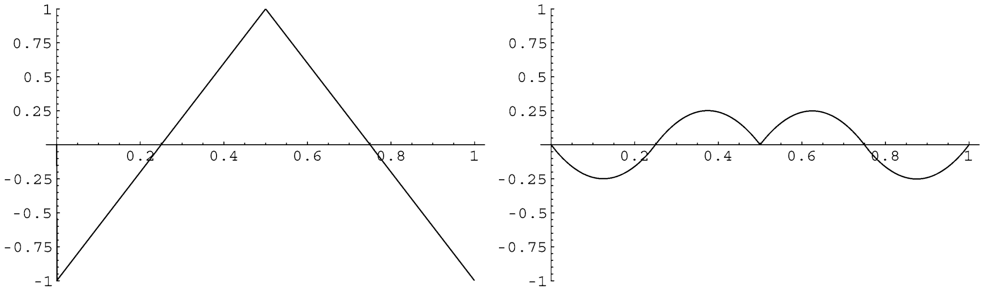

Fig. 3. $g_1(x)$ and $g_2(x)$.

$$\begin{aligned} g_{n+1}(x) = Tg_n(x) = 2\int_0^{2x} g_n(t)\,dt &= 2\int_0^{1/4} g_n(t)\,dt - 2\int_{1/4}^{2x} g_n(1/2-t)\,dt \\ &= 2\int_0^{1/4} g_n(t)\,dt - 2\int_{1/2-2x}^{1/4} g_n(t)\,dt = 2\int_0^{1/2-2x} g_n(t)\,dt \leqslant 0, \quad x \in [1/8, 1/4]. \end{aligned}$$

Using (13) for g_{n+1}, we have (14) for $n+1$ instead of n.

Next we show

$$\|g_{n+1}\|_{L^\infty} \leqslant 1/2\|g_n\|_{L^\infty}, \quad n = 1, 2, \ldots. \tag{15}$$

We note that $\operatorname{supp} g \subset [0,1]$. From (13) for g_{n+1} it follows that

$$\sup_{x\in[0,1/2]} \left|g_{n+1}(x)\right| = \sup_{x\in[1/2,1]} \left|g_{n+1}(x)\right|,$$
$$\sup_{x\in[0,1/4]} \left|g_{n+1}(x)\right| = \sup_{x\in[1/4,1/2]} \left|g_{n+1}(x)\right|.$$

Then we have $\|g_{n+1}\|_{L^\infty} = \sup_{x\in[0,1/4]} |g_{n+1}(x)|$. For $x \in [0, 1/4]$ we have

$$0 \geqslant g_{n+1}(x) = Tg_n(x) = 2\int_0^{2x} g_n(t)\,dt \geqslant 2\int_0^{1/4} g_n(t)\,dt = g_{n+1}(1/8),$$

since $g_n(t) \leqslant 0$ ($t \in [0, 1/4]$) and $g_n(t) \geqslant 0$ ($t \in [1/4, 1/2]$). Hence

$$\|g_{n+1}\|_{L^\infty} = \left|g_{n+1}(1/8)\right| = 2\left|\int_0^{1/4} g_n(t)\,dt\right| \leqslant \frac{1}{2}\|g_n\|_{L^\infty}.$$

Therefore, we obtain (15). Thus $\{u_n\}$ converge uniformly on $\mathbb{R}$. □

By Lemma 10, we have $u = Tu$. Moreover, from Lemma 9, (11) and (12) it follows that u satisfies (ii)–(v) and (vii) in Lemma 2.

3.2. Proof of Lemma 2 (Step 2)

We show that u satisfies (i) and (vi) in Lemma 2, i.e., $u \in C^\infty(\mathbb{R})$ and $u^{(k)}(0) = u^{(k)}(1) = 0$ for $k = 0, 1, 2, \ldots$ by induction.

The equality $u = Tu$, i.e.,

$$u(x) = \begin{cases} 2\int_0^{2x} u(t)\,dt, & x \in [0, 1/2], \\ 2\int_{2x-1}^{1} u(t)\,dt, & x \in (1/2, 1], \\ 0, & x \notin [0, 1] \end{cases} \tag{16}$$

implies that $u \in C^0(\mathbb{R})$ and $u(0) = u(1) = 0$.

Assume that $u \in C^k(\mathbb{R})$ and $u^{(k)}(0) = u^{(k)}(1) = 0$. Then the equality (16) implies that $u \in C^{k+1}(\mathbb{R} \setminus \{0, 1/2, 1\})$. From $u^{(k)}(0) = 0$ and the continuity of $u^{(k)}$ it follows that

$$\begin{cases} u^{(k+1)}(x) = 0 & (x < 0), \\ u^{(k+1)}(x) = (4u(2x))^{(k)} = 2^{k+2}u^{(k)}(2x) \to 0 & (x \to +0), \end{cases}$$

i.e.,

$$\lim_{x \to \pm 0} u^{(k+1)}(x) = 0.$$

By the mean value theorem, we have that

$$\frac{u^{(k)}(h) - u^{(k)}(0)}{h} = u^{(k+1)}(\theta h) \quad (0 < \theta < 1).$$

As $h \to 0$ we have that $u^{(k+1)}(0) = 0$ and $u^{(k+1)}$ is continuous at 0. By $u(x) = u(1 - x)$ we have that $u^{(k+1)}(1) = 0$ and $u^{(k+1)}$ is continuous at 1. By

$$\begin{cases} u^{(k+1)}(x) = (4u(2x))^{(k)} = 2^{k+2}u^{(k)}(2x) \to 0 & (x \to 1/2 - 0), \\ u^{(k+1)}(x) = (-4u(2x - 1))^{(k)} = -2^{k+2}u^{(k)}(2x - 1) \to 0 & (x \to 1/2 + 0), \end{cases}$$

we have that $u^{(k+1)}(1/2) = 0$ and $u^{(k+1)}$ is continuous at $1/2$. Therefore $u \in C^{k+1}(\mathbb{R})$ and $u^{(k+1)}(0) = u^{(k+1)}(1) = 0$.

3.3. *Proof of Theorem 4*

If f is a solution of (3), then the value of $f(x)$ on $[2^k, 2^{k+1}]$ is determined by the value of $f'(x)$ on $[2^{k-1}, 2^k]$.

We define a function $f \in L^\infty(\mathbb{R})$ as follows:

$$\begin{cases} f(x) = 0, & x \in (-\infty, 0), \\ f(x) = u(x), & x \in [0, 1], \\ f(x) = f'(x/2)/4, & x \in (1, 2], \\ f(x) = f'(x/2)/4, & x \in (2^k, 2^{k+1}], \quad k = 1, 2, \ldots. \end{cases}$$

By induction we show that f is a solution of (3) and that f is expressed by (4). Then from $u^{(k)}(0) = u^{(k)}(1) = 0$ for $k = 0, 1, 2, \ldots$ it follows that $u \in C^\infty(\mathbb{R})$.

First we show that

$$\begin{cases} f(x) = u(x) - u(x - 1), & x \in [0, 2], \\ f'(x) = 4f(2x), & x \in [0, 1]. \end{cases} \tag{17}$$

From $u(x) = u(1 - x)$, $u'(x) = -u'(1 - x)$ and $u(x) = u'(x/2)/4$ for $x \in [0, 1]$, it follows that, for $x \in [1, 2]$,

$$f(x) = f'(x/2)/4 = u'(x/2)/4 = -u'(1 - x/2)/4 = -u(2 - x) = -u(x - 1).$$

Then we have (17). By (17) we have

$$u'(x) = f'(x) = 4f(2x) = 4u(2x) - 4u(2x-1), \quad x \in [0,1]. \tag{18}$$

Assume that, for general $m \geqslant 2$,

$$\begin{cases} f(x) = \sum_{k=1}^{2(m-1)} (-1)^{n_k} u(x-k+1), & x \in [0, 2(m-1)], \\ f'(x) = 4f(2x), & x \in [0, m-1]. \end{cases} \tag{19}$$

Then

$$\begin{cases} f(x) = (-1)^{n_m} u(x-m+1), \\ f'(x) = (-1)^{n_m} u'(x-m+1), \end{cases} \quad x \in [m-1, m]. \tag{20}$$

If $x \in [2(m-1), 2m]$, then $x/2 \in [m-1, m]$ and $x/2 - m + 1 \in [0,1]$. By (20) and (18) we have

$$\begin{aligned} f(x) &= \frac{1}{4} f'(x/2) = \frac{1}{4}(-1)^{n_m} u'(x/2 - m + 1) \\ &= (-1)^{n_m} u\big(2(x/2 - m + 1)\big) - (-1)^{n_m} u\big(2(x/2 - m + 1) - 1\big) \\ &= (-1)^{n_m} u\big(x - (2m-1) + 1\big) - (-1)^{n_m} u(x - 2m + 1) \\ &= (-1)^{n_{2m-1}} u\big(x - (2m-1) + 1\big) + (-1)^{n_{2m}} u(x - 2m + 1). \end{aligned}$$

This shows that (19) holds for m instead of $m-1$. Therefore we have (4) and

$$f'(x) = 4f(2x), \quad x \in [0, +\infty).$$

Clearly

$$f'(x) = 4f(2x), \quad x \in (-\infty, 0).$$

Then f is a solution of (3).

4. Fourier transform of the solution f in (4)

Let $w = \chi_{[0,1]}$. Then $Tv(x) = 2(w * v)(2x)$ for $v \in L^1(\mathbb{R})$ with $\operatorname{supp} v \subset [0,1]$. Hence

$$\mathcal{F}(Tv)(\xi) = \hat{w}(\xi/2)\hat{v}(\xi/2), \qquad \hat{w}(\xi) = e^{-i\xi/2} \operatorname{sinc}\big(\xi/(2\pi)\big).$$

By

$$\mathcal{F}\big(T^{k+1}v\big)(\xi) = \hat{w}(\xi/2)\mathcal{F}\big(T^k v\big)(\xi/2),$$

we have

$$\mathcal{F}\big(T^n v\big)(\xi) = \prod_{k=1}^{n} \hat{w}\big(\xi/2^k\big) \cdot \hat{v}\big(\xi/2^n\big) = \prod_{k=1}^{n} \big(e^{-i\xi/2^{k+1}} \operatorname{sinc}\big(\xi/\big(2^{k+1}\pi\big)\big)\big) \cdot \hat{v}\big(\xi/2^n\big),$$

this is

$$\mathcal{F}\big(T^n v\big)(\xi) = e^{-i\xi(1/2 - 1/2^{n+1})} \prod_{k=1}^{n} \operatorname{sinc}\big(\xi/\big(2^{k+1}\pi\big)\big) \cdot \hat{v}\big(\xi/2^n\big). \tag{21}$$

Proof of Theorem 6. Using (21), we have

$$\hat{u_n}(\xi) = \mathcal{F}\big(T^n \chi_{[0,1]}\big)(\xi) = e^{-i\xi/2} \prod_{k=1}^{n} \operatorname{sinc}\big(\xi/\big(2^{k+1}\pi\big)\big) \cdot \operatorname{sinc}\big(\xi/\big(2^{n+1}\pi\big)\big).$$

Lemma 10 shows that $u_n = T^n \chi_{[0,1]}$ converges to u in $L^1(\mathbb{R})$. Hence $\hat{u_n}$ converges to $\hat{u}$ uniformly on $\mathbb{R}$. Using the fact that the right-hand side of (4) converges to f in $\mathcal{S}'$, we have the conclusion. □

Theorem 11. *Let $v \in L^1(\mathbb{R})$ and $\operatorname{supp} v \subset [0,1]$. Then $T^n v(x)$ converges to $\hat{v}(0)u(x)$ uniformly on $\mathbb{R}$, where u is in Lemma 10.*

Proof. By (21) and Theorem 6 we have that $\|\mathcal{F}(T^n v)\chi_{[-R,R]^c}\|_{L^1}$ is small uniformly for large $R > 0$ and that

$$\mathcal{F}\big(T^n v\big)(\xi) = \hat{u_n}(\xi)\frac{e^{i\xi/2^{n+1}}\hat{v}(\xi/2^n)}{\operatorname{sinc}(\xi/(2^{n+1}\pi))} \to \hat{u}(\xi)\hat{v}(0), \quad \text{uniformly on } [-R, R].$$

Hence $\mathcal{F}(T^n v)$ converges to $\hat{v}(0)\hat{u}(\xi)$ in $L^1(\mathbb{R})$. Therefore we have the conclusion. □

Remark 12. If v satisfies (1), (2), $\operatorname{supp} v \subset [0,1]$ and $v(1/4) = b$ ($b \in \mathbb{R}$), then $v = Tv$ and

$$b = v(1/4) = 2\int_0^{1/2} v(t)\,dt = \int_0^1 v(t)\,dt = \hat{v}(0).$$

Hence $v = bu$.

5. Proof of Theorem 7

Let $u_\delta(x) = u(x/\delta)/\delta$. Since $\int_{\mathbb{R}} u(x)\,dx = 1$,

$$v * \frac{d^k u_\delta}{dx^k} = \frac{d^k v}{dx^k} * u_\delta \to \frac{d^k v}{dx^k} \quad (\delta \to 0),$$

uniformly on each compact subset in $\mathbb{R}$ or in $L^p(\mathbb{R})$. From $u(x) = f(x)\chi_{[0,1]}(x)$ and $f'(x) = 4f(2x)$ it follows that

$$\frac{d^k u}{dx^k}(x) = 2^{(k^2+3k)/2} f\big(2^k x\big)\chi_{[0,1]}(x) = 2^{(k^2+3k)/2}(f\chi_{[0,2^k]})\big(2^k x\big).$$

Hence

$$\frac{d^k u_\delta}{dx^k}(x) = \frac{1}{\delta^{k+1}}\frac{d^k u}{dx^k}(x/\delta) = \frac{2^{(k^2+3k)/2}}{\delta^{k+1}}(f\chi_{[0,2^k]})\big(2^k x/\delta\big).$$

Let $\epsilon = \delta/2^k$. Then $\frac{d^k u_\delta}{dx^k}(x) = G_{k,\epsilon}(x)$.

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